

AUTOMORPHISM GROUPS OF POSITIVE ENTROPY ON MINIMAL PROJECTIVE VARIETIES

DE-QI ZHANG

ABSTRACT. We determine the geometric structure of a minimal projective threefold having two ‘independent and commutative’ automorphisms of positive topological entropy, and generalize this result to higher-dimensional smooth minimal pairs (X, G) . As a consequence, we give an effective lower bound for the first dynamical degree of these automorphisms of X fitting the ‘boundary case’.

1. INTRODUCTION

A normal projective variety X with only \mathbb{Q} -factorial terminal singularities is called *minimal* if the canonical divisor K_X is nef. For $G \leq \text{Aut}(X)$, the representation $G| \text{NS}_{\mathbb{C}}(X)$ on the complexified Néron-Severi group $\text{NS}_{\mathbb{C}}(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}$, is *Z-connected* if its Zariski-closure in $\text{GL}(\text{NS}_{\mathbb{C}}(X))$ is connected in the Zariski topology. We denote by $q(X) := h^1(X, \mathcal{O}_X)$ the *irregularity* of X . A birational morphism $\sigma : X \rightarrow X'$ is *crepant* if $K_X = \sigma^* K_{X'}$. We refer to [9, Definition 2.34] for the definition of terminal or canonical singularity, and to [4] for the definitions of dynamical degrees and (topological) entropy.

In this paper, we prove Theorem 1.1 below and its generalization in Theorem 1.5.

Theorem 1.1. *Let $G \leq \text{Aut}(X)$ be an automorphism group on a minimal projective threefold X with the representation $G| \text{NS}_{\mathbb{C}}(X)$ solvable and Z -connected. Then we have:*

- (1) *The null subset $N(G) := \{g \in G \mid g \text{ is of null entropy}\}$ is a normal subgroup of G such that $G/N(G) \cong \mathbb{Z}^{\oplus r}$ for some $r = r(G) \leq \dim X - 1 = 2$.*
- (2) *Suppose that $r = 2$. Then there are a G -equivariant (birational) crepant morphism $X \rightarrow X'$ and a G -equivariant finite Galois cover $\tau : A \rightarrow X'$ étale in codimension 1 for a 3-dimensional abelian variety A , with $\deg \tau \geq 2$ occurring only when $q(X) = 0$.*
- (3) *Suppose that $r = 2$ and the identity component $\text{Aut}(X)_0$ of $\text{Aut}(X)$ is trivial. Then $|N(G)| < \infty$.*

Theorem 1.1 (1) above holds in any dimension $n \geq 2$ and even for any Kähler manifold X (with $\text{NS}_{\mathbb{C}}(X)$ replaced by $H^{1,1}(X)$), i.e., $r(G) \leq n - 1$ (cf. [18, Theorem 1.2, Remark

2000 *Mathematics Subject Classification.* 32H50, 14J50, 32M05, 14J32.

Key words and phrases. automorphism group, topological entropy, Calabi-Yau variety.

The author is supported by an ARF of NUS.

1.3]; [7]). This paper is to prove Theorem 1.1 (2) (and (3)) about the geometric structure of X when $r(G) = n - 1 = 2$, and generalize it to higher dimensions (cf. Theorem 1.5).

As a consequence, we give an effective lower bound for the first dynamical degree $d_1(g)$ of $g \in G$ when $r(G) = n - 1$ (cf. Corollary 1.6).

Inspired by the main result of Dinh-Sibony [4, Theorem 1], it was asked in [18, Question 2.18] whether $r(G) = n - 1$ and (the identity component) $\text{Aut}_0(X) = (1)$ imply that the null subset $N(G) \subset G$ (as in Theorem 1.1) is a finite group. Theorem 1.1 (3) answers this question in the affirmative for minimal threefolds. When G is abelian with $r(G) = n - 1$, the finiteness of $N(G)$ is confirmed for all Kähler manifolds and in any dimension by Dinh and Sibony in [4, Theorem 1].

The motivation to consider the assumption in 1.1 is due to the Tits type result: for any $G \leq \text{Aut}(X)$ on a Kähler (resp. projective) manifold X , either $G \geq \mathbb{Z} * \mathbb{Z}$ (non-abelian free group of rank two), or G has a finite-index subgroup G_1 such that $G_1|H^{1,1}(X)$ (resp. $G_1|\text{NS}_{\mathbb{C}}(X)$) is solvable and Z -connected (see also [18, Theorem 1.1, Remark 1.3]).

Theorem 1.1 has a special consequence below, which explains why it seemed hard to construct two ‘independent and commutative’ automorphisms of positive topological entropy on a Calabi-Yau threefold except perhaps by descending those on an abelian variety to its quotients. An invertible linear map is of *positive entropy* if its spectral radius (= the maximum of modulus of eigenvalues) is greater than one.

Corollary 1.2. *Let $G \leq \text{Aut}(X)$ be an automorphism group on a minimal projective threefold X such that the representation $G^* := G|\text{NS}_{\mathbb{C}}(X) \cong \mathbb{Z}^{\oplus 2}$ and every non-trivial element of G^* is of positive entropy. Then there are a G -equivariant (birational) crepant morphism $X \rightarrow X'$ and a G -equivariant finite Galois covering $A \rightarrow X'$ étale in codimension 1 for a 3-dimensional abelian variety A .*

Remark 1.3. (1) The triviality of the identity component $\text{Aut}_0(X)$ of $\text{Aut}(X)$ is a necessary hypothesis for the finiteness of $N(G)$ in Theorem 1.1 (3). Indeed, a bigger group $G_1 := G \cdot \text{Aut}_0(X)$ satisfies also the assumption of Theorem 1.1 with $r(G_1) = r(G)$ and $N(G_1) \geq \text{Aut}_0(X)$.

(2) By the remark above and Dinh-Sibony [4, Example 4.5], for every $n \geq 2$, there is an n -dimensional abelian variety A with an automorphism group G satisfying $N(G) = \text{Aut}_0(A) \cong A$, and $G/N(G) = \mathbb{Z}^{n-1}$.

(3) As seen from the proof of Theorem 1.1, $X \rightarrow X'$ is not an isomorphism only when: A is $E_{\zeta_3}^3$ or the Jacobian of the Klein plane quartic, and the index-one Galois cover of X is a Calabi-Yau threefold and a crepant resolution of the index-one Galois cover of X' ; see Claims 2.5 and 2.11 and [15, Theorem (3.4)] for more details.

With the result above for projective threefolds, we would speculate the following:

Question 1.4. Let $G \leq \text{Aut}(X)$ be an automorphism group on a minimal projective variety X of dimension $n \geq 3$ such that the representation $G^* := G|_{\text{NS}_{\mathbb{C}}(X)} \cong \mathbb{Z}^{\oplus n-1}$ and every non-trivial element of G^* is of positive entropy. Is X then birational to the quotient of an abelian variety by a finite group?

We need to assume that $\dim X \geq 3$ in Question 1.4 because a general Wehler $K3$ surface X has Picard number two, so it is not birational to a *generalized* Kummer $K3$ surface (in the sense of T. Katsura) of necessarily Picard number ≥ 17 . Here a Wehler $K3$ is the complete intersection $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ of two hypersurfaces of bidegrees $(1, 1)$ and $(2, 2)$, where the two projections give rise to involutions ι_i so that $g := \iota_1 \circ \iota_2 \in \text{Aut}(X)$ is of topological entropy $\log(7 + 4\sqrt{3}) > 0$; I thank Serge Cantat for reminding me about the Wehler surfaces.

Below is an answer to Question 1.4 under the condition (1) or (2). The minimality of (X, G) in (1) seems achievable, and necessary too because the blowup of a variety with only canonical singularities contains a rational curve and hence has no étale torus covering.

Theorem 1.5. *Let $G \leq \text{Aut}(X)$ be an automorphism group on a smooth minimal projective variety X of dimension $n \geq 3$ such that the representation $G^* := G|_{\text{NS}_{\mathbb{C}}(X)} \cong \mathbb{Z}^{\oplus r}$ for some $r = r(G) \geq n - 1$, and every non-trivial element of G^* is of positive topological entropy. Suppose either one of the following two conditions.*

- (1) *The pair (X, G) is minimal in the sense that every G -equivariant birational morphism $X \rightarrow X'$ onto some X' with only isolated canonical singularities (and no other singularities), is an isomorphism.*
- (2) *X has no G -periodic uniruled subvariety $S (\neq \text{pt}, X)$. Here, a subvariety is G -periodic if it is stabilized by a finite-index subgroup of G .*

Then $r = n - 1$, and there is a G -equivariant finite étale Galois cover $\tau : A \rightarrow X$ for an n -dimensional abelian variety A , with $\deg \tau \geq 2$ occurring only when $q(X) = 0$.

For the G in Theorem 1.1(2) or 1.5, we obtain the following effective lower bound for the first dynamical degree $d_1(g)$ of $g \in G$. Set $1 < \sqrt{\delta_n} := \min\{|\gamma_f| > 1; \gamma_f \text{ is an eigenvalue, with the maximum modulus, of an integral polynomial } f(x) \text{ of degree } \leq 2n\}$.

Corollary 1.6. *With the notation and assumption in Theorem 1.1 or 1.5, assume that $r(G) = \dim X - 1 = n - 1 \geq 2$. Then the first dynamical degree of every $g \in G$ of positive topological entropy satisfies: $d_1(g) \geq \delta_n$.*

When $n = 2$, the result parallel to Corollary 1.6 is that $d_1(g) \geq \sqrt{\delta_{11}}$ since $\text{rank}_{\mathbb{Z}} H^2(X, \mathbb{Z}) \leq 22$ for every minimal surface X with $K_X \equiv 0$ (numerically). However, when $\dim X \geq 3$, no upper bound of $h^2(X, \mathbb{C})$ is known even for Calabi-Yau threefolds; this is related to the so called Miles Reid's fantasy: the moduli space of 3-folds X with $K_X \equiv 0$

may nevertheless be irreducible (like the case of K3 surfaces). The lower bound in Corollary 1.6 is independent of Reid's fantasy which is very hard, and hence meaningful.

When $\dim X = 2$, McMullen has proved that $d_1(g) \geq \lambda_{\text{Lehmer}} = 1.17628081 \dots$ (the Lehmer number) for all $g \in \text{Aut}(X)$ of positive topological entropy (cf. e.g. [10, p2]).

See Dinh-Sibony [4], McMullen [10] and Oguiso [13] for related results and references.

We end the introduction with examples (X, G) where $r(G) = \dim X - 1$ (in the notation of 1.5) and X is either rationally connected in the sense of Campana and Kollár-Miyaoka-Mori, or *Calabi-Yau* (i.e., $q(X) = 0$, the Kodaira dimension $\kappa(X) = 0$, and $K_X \sim_{\mathbb{Q}} 0$).

I would like to thank Keiji Oguiso for the discussion about the example below.

Example 1.7. Let $E := \mathbb{C}/(\mathbb{Z} + Z\sqrt{-1})$ and $A := E^n$ ($n \geq 1$). Then $\mu_4 = \langle \sqrt{-1} \rangle$ acts diagonally on the n -dimensional abelian variety A with a few isolated fixed points. Set $X := A/\mu_4$. As in [4, Example 4.5], a subgroup $G \cong \mathbb{Z}^{\oplus n-1}$ of $\text{SL}_n(\mathbb{Z})$ acts on A (and hence on X since the action of μ_4 is diagonal) such that every non-trivial element of G is of positive entropy on A (and hence on X ; cf. [17, Lemma 2.6] or [11, Lemma A.8]). Thus we have $r(G) = \dim X - 1$. If $n \geq 4$ (resp. $n \leq 3$) then X is a Calabi-Yau variety (resp. rationally connected); see also [1, §2], and [18, Theorem 1.2, Remark 1.3].

2. PROOF OF THEOREMS AND COROLLARIES

2.1. We use the conventions in Hartshorne's book, and [9].

For an abelian variety A , we have $\text{Aut}_{\text{variety}}(A) = T \rtimes \text{Aut}_{\text{group}}(A)$, where $\text{Aut}_{\text{variety}}(A)$ (resp. $\text{Aut}_{\text{group}}(A)$) is the group of automorphisms of A as a variety (resp. as a group), and $T = T(A) \cong A$ is the group of translations of A . By $G|X$, we mean $G \leq \text{Aut}(X)$.

2.2. Theorem 1.1 (1) follows from the argument in [18, Theorem 1.2, Remark 1.3] with X replaced by its G -equivariant resolution, $H^2(X, \mathbb{Z})$ by $\text{NS}(X)/(\text{torsion})$ and the Kähler cone by the ample cone; see also [6, Proposition 4.8, its remark].

2.3. Proof of Theorem 1.1 (2)

Since X is a projective threefold with K_X nef, the Kodaira dimension $\kappa(X) \geq 0$. By [18, Lemma 2.11], $\kappa(X) = 0$; see also 2.2. By the abundance theorem of Kawamata and Miyaoka, $IK_X \sim 0$ for some minimal $I = I(X) > 0$, called the *index* of X . Let

$$\pi : Y = \bigoplus_{i=0}^{I-1} \mathcal{O}_X(-iK_X) \longrightarrow X$$

be the *index-one covering* so that $K_Y \sim 0$. Lift G to $G \cong G|Y$. Each summand $\mathcal{O}_X(-iK_X)$ is an eigenspace of $\text{Gal}(Y/X) = \langle \sigma \rangle$ and we may assume that σ acts as the multiple by ζ_I^i on this summand, where $\zeta_I = \exp(2\pi\sqrt{-1}/I)$. So $G = G|Y$ normalize $\langle \sigma \rangle$ (for later use).

The identification $G|X$ with $G|Y$ also identifies $N(G|X)$ with $N(G|Y)$; see [6, Proposition 4.8, its remark], so $G/N(G) = \mathbb{Z}^{\oplus 2}$ holds on X and also on Y .

Definition 2.4. As in [16], for a normal projective variety X of dimension $n \geq 3$ and with only canonical singularities, take a resolution $\iota : Z \rightarrow X$ crepant in codimension two (or minimal in codimension two), e.g., a composite of a terminalization (cf. [2, Corollary 1.4.3]) and a desingularization. Then we define the (multi) *linear form* $c_2(X)$ on $N^1(X) \times \cdots \times N^1(X)$ (with $N^1(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$) as

$$H_1 \cdots H_{n-1} \cdot c_2(X) := \iota^* H_1 \cdots \iota^* H_{n-2} \cdot c_2(Z).$$

By Hironaka's equivariant resolution, when $G \leq \text{Aut}(X)$ is given, we may also choose ι to be G -equivariant, so that $g^* c_2(X) = c_2(X)$ for all $g \in G$.

We remark that in the case where K_X is nef, $c_2(X) = 0$ (as a linear form) holds if and only if $H^{n-2} \cdot c_2(X) = 0$ for one ample divisor H , by Miyaoka's pseudo-effectivity of c_2 of every terminalization of X and since $N^1(X)$ is spanned by ample divisors H_i (and noting that $H - \varepsilon H_i$ is an ample \mathbb{Q} -divisor for small ε).

Claim 2.5. (1) *If $Z \in \{X, Y\}$, then $H_Z \cdot c_2(Z) = 0$ for a nef and big \mathbb{R} -divisor H_Z .*

(2) *If $q(X) > 0$ (resp. $q(Y) > 0$), then X (resp. Y) is an abelian variety, so Theorem 1.1(2) is true.*

(3) *If $c_2(X) = 0$ or $c_2(Y) = 0$ as linear form, then Theorem 1.1(2) is true.*

We prove Claim 2.5 (2) first. We may assume that the s -th group $G^{(s)}$ in the derived series of G satisfies $G^{(s)}| \text{NS}_{\mathbb{R}}(X) = \text{id}$. Take a finite-index subgroup $G_1 \leq G$ such that $G_1| \text{NS}_{\mathbb{C}}(Y)$ is Z -connected and $G_1/N(G_1) \cong \mathbb{Z}^{\oplus 2}$ still holds on X and Y . Take an ample Cartier divisor $M \subset X$ and write $M' = \pi^* M$. Then the class $[M'] \in \text{NS}_{\mathbb{R}}(Y)$ is $G_1^{(s)}$ -invariant. Now a result of David Lieberman (and Hironaka's equivariant resolution) imply that $\text{Aut}_{M'}(Y) := \{g \in \text{Aut}(Y) \mid g^* M' \equiv M' \text{ in } \text{NS}_{\mathbb{R}}(Y)\}$ is a finite-index over-group of $\text{Aut}_0(Y)$ (where the latter acts trivially on $\text{NS}_{\mathbb{R}}(Y)$); see [17, Lemma 2.23]. So $G_1^{(s)}| \text{NS}_{\mathbb{C}}(Y)$ is a finite group and hence is trivial because $G_1^{(s)}| \text{NS}_{\mathbb{C}}(Y)$ is also Z -connected. Thus, $G_1| \text{NS}_{\mathbb{C}}(Y)$ is solvable and also Z -connected with $r(G_1|Y) = 2$.

By [18, Lemma 2.13] (see also 2.2), either $q(Y) = 0$, or the albanese map $\text{alb}_Y : Y \rightarrow A := \text{Alb}(Y)$ is a well defined morphism (for Y having only rational singularities and using [5, Lemma 8.1]) and is birational and surjective. In the latter case, $Y \cong A$ (and Theorem 1.1 (2) is true as in (3) below), because $K_Y \equiv 0 \equiv K_A$, and alb_Y is neither a small contraction nor crepant since A is smooth. The same argument applies to X . This proves (2).

(1) We treat only Y , since the case X is similar and simpler. As in (2), replacing G with its finite-index subgroup, we may assume that $G| \text{NS}_{\mathbb{C}}(Y)$ is solvable and Z -connected.

We use the argument in the proof of [6, Lemma 5.2]. In particular, there is a common nef eigenvector $0 \neq L \in \text{NS}_{\mathbb{R}}(Y) \cap c_2(Y)^\perp$ for G on Y . For $g \in G$, write $g^*L \equiv \chi(g)L$ with $\chi(g) \in \mathbb{R}_{>0}$. Consider the homomorphism below

$$f : G|Y \longrightarrow \mathbb{R}, \quad g \mapsto \log \chi(g).$$

We have $N(G) \subseteq \text{Ker}(f)$. Consider the case $N(G) \neq \text{Ker}(f)$. Take $g \in \text{Ker}(f) \setminus N(G)$. By the generalized Perron-Frobenius theorem [3], there are nef \mathbb{R} -divisors L_g^\pm such that $(g^\pm)^*L_g^\pm \equiv d_1(g^\pm)L_g^\pm$. Here $d_1(h)$ denotes the first dynamical degree of $h \in \text{Aut}(Y)$. We have $L.L_g^+.L_g^- \neq 0$ by [4, Lemma 4.4], so $H_Y := L + L_g^+ + L_g^-$ is nef and big with $H_Y.c_2(Y) = 0$. Indeed, $H_Y^3 \geq L.L_g^+.L_g^- > 0$; also $L.c_2(Y) = 0$ by the choice of L , and $L_g^\pm.c_2(Y) = (g^\pm)^*L_g^\pm.(g^\pm)^*.c_2(Y) = d_1(g^\pm)L_g^\pm.c_2(Y)$, so $L_g^\pm.c_2(Y) = 0$, since $d_1(g^\pm) > 1$ for $g \notin N(G)$.

Thus we may assume that $\text{Ker}(f) = N(G)$. If every $\chi(g)$ is 1, or $d_1(g)$, or $1/d_1(g^{-1})$, then $\text{Im}(f)$ is discrete in \mathbb{R} by [4, Corollary 2.2]; so $\mathbb{Z}^{\oplus 2} \cong G/N(G) \cong \text{Im}(f) = \mathbb{Z}^{\oplus s}$ with $s \leq 1$, absurd. Therefore, we may assume that $\chi(g) \neq 1$, $d_1(g)$, or $1/d_1(g^{-1})$ for some $g \in G$. Then we have $L.L_g^+.L_g^- \neq 0$ by [4, Lemma 4.4]. Thus Claim 2.5 (1) is true as above by taking $H_Y := L + L_g^+ + L_g^-$.

(3) If $c_2(Z) = 0$ as a linear form for some $Z \in \{X, Y\}$, then $Z = A/H$ for an abelian variety A and $H \leq \text{Aut}_{\text{variety}}(A)$ acting on A freely in codimension two (cf. [16, Cor, p.266]). Replacing A , we may assume $A \rightarrow X$ is (Galois and) the Albanese closure in codimension one in the sense of [12, Lemma 2.12] (Kawamata's characterization of abelian variety may also be used), so that G lifts to $G \cong G|A \leq \text{Aut}_{\text{variety}}(A)$. This proves (3) and Claim 2.5.

We resume the proof of Theorem 1.1 (2). By Claim 2.5, we may assume that $q(Z) = 0$ and $c_2(Z) \neq 0$ for $Z = X, Y$. By [15, Theorem (3.4)], Y is smooth and there are a birational crepant morphism $Y \rightarrow Y'$ (the unique contraction of all divisors perpendicular to $c_2(Y)$) and an étale-in-codimension-two Galois covering $A \rightarrow Y'$ from an abelian variety A of dimension three. Further, either $A = E_{\zeta_3}^3$ (the product of three copies of the elliptic curve of period $\zeta_3 = \exp(2\pi\sqrt{-1}/3)$) and $|\text{Gal}(A/Y')|$ divides 27, or A is the Jacobian of the Klein plane quartic $\{X_0X_1^3 + X_1X_2^3 + X_2X_0^3 = 0\}$ and $\text{Gal}(A/Y') \cong \mathbb{Z}/(7)$. Note that $(G|Y).\langle\sigma\rangle$ (with $\langle\sigma\rangle = \text{Gal}(Y/X)$ normalized by $G|Y$ as mentioned early on) descends to a regular action on Y' and we let $Y' \rightarrow X' := Y'/\langle\sigma\rangle$ be the quotient map. Then there is a birational morphism $X \rightarrow X'$ such that the two natural composites below coincide: $Y \rightarrow Y' \rightarrow X'$, $Y \rightarrow X \rightarrow X'$. Our $G|X$ (and $G|Y'$) descend to a regular action $G|X' \cong G$. Since $Y \rightarrow X$ is étale in codimension one, so is $Y' \rightarrow X'$; further, $X \rightarrow X'$ is crepant because so is $Y \rightarrow Y'$. Therefore, X' has only canonical singularities (and \mathbb{Q} -factorial because $A \rightarrow X'$ is finite) with $K_{X'} \sim_{\mathbb{Q}} 0$. Note that the composite $A \rightarrow Y' \rightarrow X'$ is étale in codimension one. Replacing A , we may assume that $A \rightarrow X'$ is (Galois and) the Albanese

closure in codimension one in the sense of [12, Lemma 2.12], so that $G \cong G|X'$ lifts to $G \cong G|A \leq \text{Aut}_{\text{variety}}(A)$. This proves Theorem 1.1 (2).

2.6. Proof of Theorem 1.1 (3)

The identification of $G|X$, $G|X'$ and $G|A$ also identifies $N(G|X)$, $N(G|X')$ and $N(G|A)$; see [6, Proposition 4.8, its remark], so $G/N(G) = \mathbb{Z}^{\oplus 2}$ holds on X , X' and A . Let G_0 be a finite-index normal subgroup of G such that $G_0|NS_{\mathbb{C}}(A)$ is Z -connected. Since $NS_{\mathbb{C}}(X')$ pulls back to a subspace of $NS_{\mathbb{C}}(A)$, our $G_0|NS_{\mathbb{C}}(X')$ is also Z -connected. Further, $G_0|NS_{\mathbb{C}}(X)$ is Z -connected, since $NS_{\mathbb{C}}(X)$ is spanned by the pullback of $NS_{\mathbb{C}}(X')$ and finitely many irreducible components in the exceptional locus of $X \rightarrow X'$ which is stable under the actions of $G|X$ and $G_0|X$. Note that $N_0 := G_0 \cap N(G)$ equals $N(G_0)$ on all of X , X' and A . We also have $G_0/N_0 \cong \mathbb{Z}^{\oplus 2}$ and $r(G_0) = 2$.

Claim 2.7. *$G_0|NS_{\mathbb{C}}(A)$ is solvable. If N_0 is finite, then $N(G)$ is finite.*

We prove Claim 2.7. The first part follows from the proof of Claim 2.5 and the solvability of the action $G_0 (\leq G)$ on $NS_{\mathbb{C}}(X)$ and hence on $NS_{\mathbb{C}}(X')$. Suppose $s_2 := |N_0| < \infty$. Set $s_1 := |G : G_0|$ and $s = s_1 s_2$. Take any $n \in N(G)$. Then $n^{s_1} \in G_0 \cap N(G) = N_0$ and hence $n^s = \text{id}$. Thus $N^* := N(G)|NS_{\mathbb{C}}(X)$ is a periodic group with bounded exponent, so it is a finite group by Burnside's theorem. Hence $N(G) \leq \text{Aut}_M(X)$, where $M := \sum_{n^* \in N^*} n^* M'$ with an ample divisor $M' \subset X$. Now $N(G)$ is finite by the assumption $\text{Aut}_0(X) = (1)$ and Lieberman's result as in Claim 2.5. This proves Claim 2.7.

We resume the proof of Theorem 1.1 (3). By Claim 2.7 and replacing G with G_0 , we may assume that $G|NS_{\mathbb{C}}(A)$ is already Z -connected (and also solvable). Let \bar{G} and \bar{N} be the images of G and $N(G)$ under the projection $\text{Aut}_{\text{variety}}(A) = T \rtimes \text{Aut}_{\text{group}}(A) \rightarrow \text{Aut}_{\text{group}}(A)$, where $T = \text{Aut}_0(A)$ is the group of translations.

We are going to apply Oguiso [14, Lemma 2.5]. Consider the faithful matrix representation $\bar{G}|H^0(A, \Omega_A^1)$. We say that $g \in \bar{G}$ is *unipotent* if so is its representation matrix, and let U be the set of unipotent elements in \bar{G} . An element g of \bar{G} is in \bar{N} if and only if all eigenvalues of its matrix representation are of modulus 1, i.e., they are of roots of 1 by Kronecker's theorem (and with bounded minimal polynomial over \mathbb{Q}). Thus $U \subseteq \bar{N}$ and there is an $s > 0$ such that $n^s \in U$ for all $n \in \bar{N}$. If every element of \bar{N} is periodic then these periods divide s ; thus, by Burnside's theorem, $\bar{N}|H^{1,1}(A)$ is a finite group, so $N(G)|H^{1,1}(A)$ and hence $N(G)|NS_{\mathbb{C}}(X)$ (embedded in the former by the pullback) and even $N(G)$ are all finite groups (cf. the proof of Claim 2.7).

Therefore, we may assume that U contains an element of infinite order. In particular, the pointwise fixed set $B := A^U = \{a \in A \mid u(a) = a, \forall u \in U\}$ satisfies $0 \leq B < A$. By

[14, Lemma 2.5], U is a normal subgroup of \bar{N} (and also of \bar{G} because conjugate action preserves the unipotent property). It is known then that $U|H^0(A, \Omega_A^1)$ can be regarded as a subgroup of the unipotent group $U(q(A), \mathbb{C})$ and of the upper triangular group $T(q(A), \mathbb{C})$, so the matrix representation of U has a common eigenvector (corresponding to the unique eigenvalue 1). Thus $B = \cap_{u \in U} \text{Ker}(u - \text{id})$ satisfies $0 < B < A$. Since U is normal in \bar{G} , our subtorus B is \bar{G} -stable. So G permutes the cosets A/B . Thus we have an induced G -equivariant fibration $A \rightarrow A/B$. Therefore, $2 = r(G) \leq \dim A - 2 = 1$ by the proof of [18, Lemma 2.10] (see also 2.2). This is a contradiction. This proves Theorem 1.1 (3).

2.8. Proof of Corollary 1.2

Let $G_1 \leq G$ be the inverse of the identity connected component for the Zariski-closure of G^* in $\text{GL}(\text{NS}_{\mathbb{C}}(X))$. Then G_1 satisfies the hypothesis of Theorem 1.1 and $r(G_1) = r(G) = 2$, since $|G : G_1| < \infty$ and $\text{Ker}(G \rightarrow G^*) = N(G)$ by the assumption. Thus Corollary 1.2 holds. Indeed, the G -equivariant property is true because the morphisms involved in the proof of Theorem 1.1, like the index-one cover, or Albanese-closure in codimension one, or the c_2 -birational contraction in [15, Theorem (3.4)] are all canonical.

2.9. Proof of Theorem 1.5

At first, we assume no smoothness of X , but assume X has only canonical singularities. Note that $r = r(G) = n - 1$ (cf. 2.2, [4, Theorem 4.7], [18, Theorem 1.2, Remark 1.3]).

Applying [4, Theorems 4.7 and 4.3] to a G -equivariant resolution of X , there are nef \mathbb{R} -divisors L_1, \dots, L_n as common eigenvectors of G such that $L_1 \cdots L_n \neq 0$ and the homomorphism below is an isomorphism onto a spanning lattice (where we write $g^*L_i \equiv \chi_i(g)L_i$):

$$f : G | \text{NS}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}^{n-1}, \quad g \mapsto (\log \chi_1(g), \dots, \log \chi_{n-1}(g)).$$

Since $L_1 \cdots L_n = g^*(L_1 \cdots L_n) = \chi_1(g) \cdots \chi_n(g)L_1 \cdots L_n$ we have $\chi_1 \cdots \chi_n = 1$. Set $H := \sum_{i=1}^n L_i$. Since $H^n \geq L_1 \cdots L_n > 0$, our H is nef and big.

Claim 2.10. $H^{n-1}.K_X = 0$, and $H^{n-2}.c_2(X) = 0$.

We prove Claim 2.10. Take $M := L_1^{i_1} \cdots L_n^{i_n}$ with $\sum_{k=1}^n i_k = s$. When calculating $M.c_p(X)$ for $p = 1$ or $p = 2$, we let $s = n - p$. For $g \in G$, we have $g^*M = e(g)M$ with $e(g) = \chi_1(g)^{i_1} \cdots \chi_n(g)^{i_n}$. Since $M.c_p(X) = g^*M.g^*c_p(X) = e(g)M.c_p(X)$ and since H^s is a combination of such M , it suffices to show that $e(g) \neq 1$ for some $g \in G$ (so that $M.c_p(X) = 0$). Suppose the contrary that $e(g) = 1$ for all $g \in G$. Taking log and using $\chi_1 \cdots \chi_n = 1$, we have $(i_1 - i_n) \log \chi_1 + \cdots + (i_{n-1} - i_n) \log \chi_{n-1} = 0$ on G . Since the image of the homomorphism f above is a spanning lattice, this happens only when $i_1 - i_n = \cdots = i_{n-1} - i_n = 0$. Thus $n - 1 \geq s = \sum_{k=1}^n i_k = ni_1$, so $i_1 = 0$ and hence $s = 0$, absurd. This proves Claim 2.10.

Since $H^{n-1}.K_X = 0$ and K_X is nef, we have $K_X \equiv 0$ by [12, Lemma 2.2]. So $K_X \sim_{\mathbb{Q}} 0$ by [5, Theorem 8.2]. By the \mathbb{R} -divisor version [2, Theorem 3.9.1] of Kawamata's base point freeness theorem, there is a birational morphism $\gamma : X \rightarrow X'$ such that $H = \gamma^*H'$ for an ample \mathbb{R} -divisor H' . The result below should be well known, but we prove it for the lack of reference. As said early, every singularity of X is assumed to be canonical only.

Claim 2.11. *$\gamma : X \rightarrow X'$ is crepant, and hence X' has only canonical singularities. Further, the indices of X and X' coincide and are denoted as J so that $JK_X \sim 0$ and $JK_{X'} \sim 0$.*

We prove Claim 2.11. Let $I := I(X)$ be the index of X and

$$\pi : Y = \text{Spec} \oplus_{i=0}^{I-1} \mathcal{O}_X(-iK_X) \longrightarrow X$$

the index-one cover which is étale in codimension one, where $K_Y \sim 0$. Let $Y \rightarrow Y' \rightarrow X'$ be the Stein factorization of $Y \rightarrow X \rightarrow X'$. Then $Y' \rightarrow X'$ is étale in codimension one because so is $Y \rightarrow X$. By [9, Proposition 5.20], every singularity of Y is canonical and hence rational. Applying Kollár's torsion freeness result for higher direct image of dualizing sheaf to the *composite* of $Y \rightarrow Y'$ and a resolution of Y , the first condition in [8, Proposition 3.12] is satisfied by this composite because $\mathcal{O}(K_Y) \cong \mathcal{O}_Y$ now (cf. also [9, Corollary 2.68]), so Y' has only rational singularities. Applying [9, Lemma 5.12] to the above composite, we have $\mathcal{O}_{Y'}(K_{Y'}) \cong \mathcal{O}_{Y'}$. Thus Y' has only Gorenstein canonical singularities, so X' has only log terminal singularities by [9, Proposition 5.20], and $IK_{X'} \sim 0$; further, $Y \rightarrow Y'$ is crepant because $K_Y \sim 0 \sim K_{Y'}$. The commutative diagram in the Stein factorization above then implies that $X \rightarrow X'$ is crepant. The last part of Claim 2.11 is true because $I(X')K_{X'} \sim 0$ implies $I(X')K_X \sim 0$ by considering the fibre product over X' , of $X \rightarrow X'$ and the index-one covering of X' . This proves Claim 2.11.

$(H')^{n-2}.c_2(X') = H^{n-2}.c_2(X) = 0$ implies $c_2(X') = 0$ as a linear form (cf. 2.4). Our $\gamma : X \rightarrow X'$ is G -equivariant, because a curve $C \subset X$ is contracted by γ if and only if $H.C = 0$; if and only if $L_i.C = 0$ (for all i , since L_i is nef); if and only if $L_i.g_*C = 0$ (since L_i is semi G -invariant); if and only if g_*C is contracted by γ . This argument and [9, Remark (2), p46] show that $L_i = \gamma^*L'_i$ for some nef L'_i on X' (with $g^*L'_i \equiv \chi_i(g)L'_i$). Thus, $H' = \sum_{i=1}^n L'_i$.

We assert that X' has no G -periodic subvariety S' of dimension $s \in \{1, \dots, n-1\}$. Indeed, the proof of Claim 2.10 applied to a finite-index subgroup of $G|X'$, would show that $(H')^s.S' = 0$, contradicting the ampleness of H' .

If γ is not an isomorphism, then the exceptional locus $\text{Exc}(\gamma)$ is non-empty whose irreducible components E_i are permuted by G and hence G -periodic. So $\gamma(E_i)$ are G -periodic and hence a point by the assertion above. Thus E_i is covered by fibres of γ and hence uniruled, since every fibre of a partial resolution of a log terminal singularity is rationally chain connected by Hacon-McKernan's solution to Shokurov's conjecture.

We remark that for every G -periodic subvariety $\text{pt} \neq S \subset X$, the image $\gamma(S) \subset X'$ is G -periodic and hence a point by the assertion above, thus $S \subseteq \text{Exc}(\gamma)$.

By either conditoin, we may assume $\gamma = \text{id}$. As in Theorem 1.1(2), Theorem 1.5 follows from the vanishing of $c_i(X)$ ($i = 1, 2$), Bogomolov decomposition [1] and *smoothness* of X .

2.12. Proof of Corollary 1.6

It follows from that $\text{rank}_{\mathbb{Z}} H^1(A, \mathbb{Z}) = 2n$ and $d_1(g) = d_1(g_A)$ (cf. [17, Lemma 2.6] or [11, Lemma A.8]); here g_A on A is the lifting of g .

REFERENCES

- [1] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, *Classification of Algebraic and Analytic Manifolds* (Katata, 1982, ed. K. Ueno), Progr. Math., **39** Birkhäuser 1983, pp. 1–26.
- [2] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, *J. Amer. Math. Soc.* **23** (2010) 405 – 468.
- [3] G. Birkhoff, Linear transformations with invariant cones, *Amer. Math. Month.* **74** (1967) 274–276.
- [4] T.-C. Dinh, N. Sibony, Groupes commutatifs d’automorphismes d’une variété kählérienne compacte, *Duke Math. J.* **123** (2004) 311–328.
- [5] Y. Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, *J. Reine Angew. Math.* **363** (1985), 1–46.
- [6] J. Keum, K. Oguiso and D. -Q. Zhang, Conjecture of Tits type for complex varieties and theorem of Lie-Kolchin type for a cone, *Math. Res. Lett.* **16** (2009), no. 1, 133–148.
- [7] J. H. Kim, Solvable automorphism groups of a compact Kaehler manifold, arXiv:**0712.0438v7**.
- [8] J. Kollár, Higher direct images of dualizing sheaves, II, *Ann. of Math.*, **124** (1986), 171–202.
- [9] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., **134**, Cambridge Univ. Press, 1998.
- [10] C. McMullen, K3 surfaces, entropy and glue, preprint of 7 October 2009, at: <http://www.math.harvard.edu/~ctm/papers/index.html>
- [11] N. Nakayama, D. -Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds, *Proc. London Math. Soc.*, **99** (2009) 725 – 756.
- [12] N. Nakayama and D. -Q. Zhang, Polarized endomorphisms of complex normal varieties, *Math. Ann.*, **346** (2010) 991–1018.
- [13] K. Oguiso, The third smallest Salem number in automorphisms of K3 surfaces, arXiv:**0905.2396**.
- [14] K. Oguiso, Automorphisms of hyperkähler manifolds in the view of topological entropy, *Algebraic geometry*, pp. 173–185, Contemp. Math., Vol. **422**, Amer. Math. Soc., Providence, RI, 2007.
- [15] K. Oguiso and J. Sakurai, Calabi-Yau threefolds of quotient type, *Asian J. Math.* **5** (2001), 43–77.
- [16] N. I. Shepherd-Barron and P. M. H. Wilson, Singular threefolds with numerically trivial first and second Chern classes, *J. Alg. Geom.* **3** (1994), 265–281.
- [17] D. -Q. Zhang, Dynamics of automorphisms on projective complex manifolds, *J. Diff. Geom.* **82** (2009) 691–722.
- [18] D. -Q. Zhang, A theorem of Tits type for compact Kähler manifolds, *Inven. Math.* **176** (2009), 449–459.

NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2, SINGAPORE 117543

E-mail address: `matzdg@nus.edu.sg`